

The frequency spectrum of the flutes

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As one type of lip-driven wind instruments, although flutes have a very long history, the acoustics of flutes is still unclear. Here, using the admittance matching method, the resonance frequencies of different tones for a flute are calculated with the precise correction for geometrical sizes. The result shows that the given frequency spectra of the flute are almost the same as the designed values.

Keywords: Lip-wind-instrument; admittance matching; resonance frequency; frequency spectrum.

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1. Introduction

Lip-driven wind instruments have a very long history,^{1,2} dating back to those made from hollow plant stems, sea shells, and animal horns.^{3,4} Early instruments usually had a more or less conical bore derived from sea shells or animal bores. Exception arose in the case of cylindrical main pipe instruments derived from plant stems with side holes hollowed out by the action of the fire.⁵ In order to produce more melodic possibilities and an octave of complete diatonic scale, the development on wind instruments was to increase greatly the length of the main pipe and to narrow its side hole. In this way, more pitches of the desired tones can be obtained. After a long period of evolution and development, the flute has been essentially improved. So far, it has become one of the most popular lip-driven wind instruments.

Most flutes are made of bamboo, but can come in jade, metal, and other materials. In general, a flute is a cylindrical pipe with several side holes with designed arrangement in positions. One end is closed by a cork plug and the other end open to

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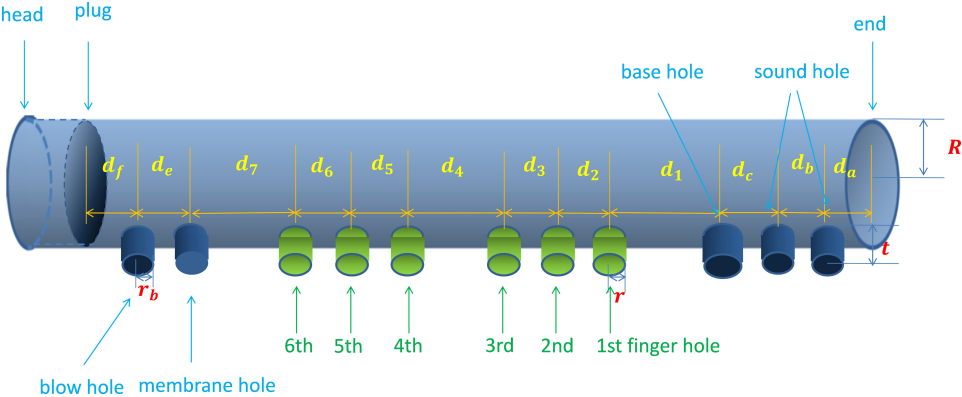


Fig. 1. (Color online) Schematic picture of the flute. The flute consists of three pieces, including head-joint, middle-joint and foot.

the atmosphere. A flute can be divided into three parts, the head-joint, the middle-joint and the foot, which is shown in Fig. 1. The head-joint part includes the blow hole (mouth-hole) and membrane hole (Chinese flute), the middle-joint part includes finger holes, and the foot-joint part forms the rest of the flute. Different flute geometries, for example, different head-joint and foot-joint lengths and side hole areas, cause different tone ranges (i.e. resonance frequency ranges) of sound wave. The key of a flute is defined by the frequency of sound emitted by opening the first three tone holes (labeled as in Fig. 1). The so-called *C, D, E, F, G, A, B* key flutes, corresponding, respectively, to the frequencies 440, 494, 554, 587, 659, 740, 831 Hz, can be made by changing the length of the main pipe or the shapes of the side holes.

Beautiful melody of a flute is controlled by players' active control. Basic disposition principle of finger holes in all wind instruments involves three fingers of each hand, where the left hand area is close to the mouthpiece. These six finger holes provide enough flexibility to produce a diatonic major scale of good tuning. All-closed fingering denotes its lowest tone of the flute, i.e. tonic. Here, we list a popular fingering system in Fig. 2, where bass 5 is the tonic. The resonance frequencies of the flutes with different tones have been discussed in Ref. 6, where the frequency formula of finite open tube and the empirical main pipe length modification are used. However, such approach lacks physical understanding and largely depends on playing experience. Here, the admittance matching method is adopted, which provides precise analytic solutions of the flute frequency spectra.

This paper is organized as follows: In Sec. 2, we employ the inverse surface Green's function to formulate the end admittances for complex structures. Thus, the relationship between resonance frequency and admittance matching of the combined pipe is provided. Then, we introduce a theoretical model to calculate the resonance frequency of the flute in Sec. 3. We interpret an appropriate correction method in Sec. 4, which characterizes the effect of open ends as well as the side holes. By applying the admittance matching method with correction principles, the resonance

tones	Finger holes:					
	6th	5th	4th	3rd	2nd	1st
Bass 5	●	●	●	●	●	●
Bass 6	●	●	●	●	●	○
Bass 7	●	●	●	●	○	○
Alto 1	●	●	●	○	○	○
Alto 2	●	●	○	○	○	○
Alto 3	●	○	○	○	○	○
Alto 4(1)	●	○	○	○	○	○
Alto 4(2)	○	●	●	○	○	○
Alto 5(1)	●	●	●	●	●	●
Alto 5(2)	○	●	●	●	●	●
Alto 6	●	●	●	●	●	○
Alto 7	●	●	●	●	○	○
High 1	●	●	●	○	○	○
High 2	●	●	○	○	○	○
High 3	●	○	○	○	○	○
High 4(1)	●	○	○	○	○	○
High 4(2)	○	●	●	●	●	○
High 5(1)	○	●	●	●	●	●
High 5(2)	○	●	●	○	○	○
High 6	●	●	○	●	●	○
High 7	●	●	○	●	●	●

Fig. 2. (Color online) The specific fingering system of the flute, defining the lowest tone of the flute (with all the finger holes are closed) is bass 5. Solid circles (●) represent closed finger holes, hollow circles (○) represent open finger holes and there is a horizontal line inside the circles (◐) which represent the half-open and half-closed finger holes. Especially, four tones including alto 4, alto 5, high 4 and high 5, have two different fingerings.

frequency spectrum of the flute in C is obtained in Sec. 5, which agrees well with the data provided by the manufacturer. The conclusions are summarized in Sec. 6.

2. End Admittances and Resonance Condition for a Combined Pipe

For an infinite cylindrical tube with cross-section area S , it is well known that the behaviors of the acoustic wave propagating are determined by the two characteristic parameters, the wave number of the acoustic wave, $k = \frac{\omega}{c}$, and the admittance of the tube, $Y_0 = \frac{S}{\rho c}$. Here, ρ is the density of air and c the velocity of sound in air, ω is the angular frequency of the acoustic wave. Obviously, the wave number k is independent of the structure of pipe. For an open/closed-end tube with a finite length d , its end admittance and its resonance condition can be calculated by different methods.⁷ As its both ends are open or closed, the input admittance is $Y^{o-o} = Y^{c-c} = -Y_0 \cot(kd)$. Since the resonance effect appears at the end admittance being infinity, the resonance frequencies can be obtained as $\omega^{o-o} = \omega^{c-c} = \frac{n\pi c}{d}$. For one closed



Fig. 3. (Color online) Schematic of a single pipe modeled as a combined pipe.

end, the input admittance is $Y^{o-c} = Y_0 \tan(kd)$ with resonance frequencies, $\omega^{o-c} = \frac{(2n-1)\pi c}{2d}$.

As the system is combined by many pipes to form a network, a few methods have the ability to give its admittances. The interface response theory whose foundation is the surface Green's function has natural advantages.⁸⁻¹⁰ The inverse surface Green's function of one finite tube with length d has the form

$$g^{-1} = \begin{pmatrix} -Y_0 \cot(kd) & \frac{Y_0}{\sin(kd)} \\ \frac{Y_0}{\sin(kd)} & -Y_0 \cot(kd) \end{pmatrix}. \quad (1)$$

Consider the pipe as a series of two cylindrical tubes with respective length d_1 and d_2 with identical cross-sectional area S (shown in Fig. 3). We can treat tube 2 as a load with admittance $Y_0 \cot(kd_2)$, connected to the end of tube 1. For such a combined pipe, the inverse surface Green's function of tube 1 is straightly obtained as

$$g_I^{-1} = \begin{pmatrix} -Y_0 \cot(kd_1) & \frac{Y_0}{\sin(kd_1)} \\ \frac{Y_0}{\sin(kd_1)} & -Y_0 \cot(kd_1) - Y_0 \cot(kd_2) \end{pmatrix}. \quad (2)$$

We directly get the end admittance at cross-section I,

$$Y_I = \frac{1}{g_I(1,1)} = -Y_0 \cot(kd_1 + kd_2). \quad (3)$$

Similarly, the inverse surface Green's function of tube 2 has the form

$$g_{II}^{-1} = \begin{pmatrix} -Y_0 \cot(kd_2) & \frac{Y_0}{\sin(kd_2)} \\ \frac{Y_0}{\sin(kd_2)} & -Y_0 \cot(kd_1) - Y_0 \cot(kd_2) \end{pmatrix}.$$

Then, we get

$$Y_{II} = \frac{1}{g_{II}(1,1)} = -Y_0 \cot(kd_1 + kd_2).$$

It is obvious that the end admittance of the simple pipe can be expressed directly as $-Y_0 \cot[k(d_1 + d_2)]$, thus we have validated the algorithm of end admittance through Green's function.

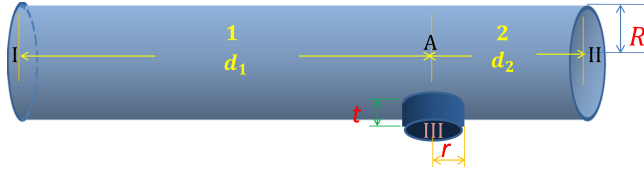


Fig. 4. (Color online) Schematic diagrams of a cylindrical pipe with a side hole.

As the most basic unit in case of the flute, we consider a cylindrical pipe with a single side branch mounted on its shell of side wall as shown in Fig. 4. The radii of the main pipe and the side hole are R and r , respectively. The geometrical length of the branch is t . “A” represents the position associated to the main pipe, where the branch is mounted. The pipe is separated by a side hole into two parts, tube 1 and tube 2, with different lengths (d_1 and d_2) and the same cross-section (πR^2).

In this case, we treat tube 1 and the side hole as two loads connected to the same end of tube 2. Looking at the cross-section II, only the inverse Green’s function element (2, 2) of g_{II}^{-1} is changed to $-Y_0 \cot(kd_2) - Y'_0 \cot(kt) - Y_0 \cot(kd_1)$. We write

$$g_{\text{II}}^{-1} = \begin{pmatrix} -Y_0 \cot(kd_2) & \frac{Y_0}{\sin(kd_2)} \\ \frac{Y_0}{\sin(kd_2)} & -Y_0 \cot(kd_2) - Y'_0 \cot(kt) - Y_0 \cot(kd_1) \end{pmatrix}. \quad (4)$$

Further, the end admittance of such combined pipe at cross-section II is

$$Y_{\text{II}} = \frac{1}{g_{\text{II}}(1, 1)} = \frac{-Y_0^2 + Y_0 Y'_0 \cot(kd_2) \cot(kt) + Y_0^2 \cot(kd_2) \cot(kd_1)}{-Y_0 \cot(kd_2) - Y'_0 \cot(kt) - Y_0 \cot(kd_1)}. \quad (5)$$

Generally, we only discuss the resonance occurring at the side holes’ positions, which is significant for the flute investigations. At cross-section A, the end admittances of tube “1”, “2” and the side hole can be expressed, respectively. By quoting the resonant condition that the total admittance at the cross-section A is added up to zero,¹¹ we get

$$Y_A^1 + Y_A^2 + Y_A^S = 0. \quad (6)$$

For an open side hole, the resultant resonance condition is

$$\cot(kd_1) + \cot(kd_2) = -\frac{r^2}{R^2} \cot(kt). \quad (7)$$

For a closed side hole, the end admittance of closed side hole at cross-section A is $Y_A^{C-S} = Y'_0 \tan(kt)$. So the resultant resonance condition is

$$\cot(kd_1) + \cot(kd_2) = \frac{r^2}{R^2} \tan(kt). \quad (7')$$

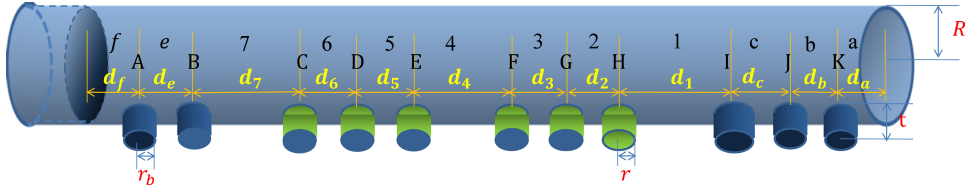


Fig. 5. (Color online) Schematic diagram of the flute on playing condition of bass 6.

3. The Resonance Frequency Calculation Method of the Flutes

In this section, we will calculate the resonance frequencies of the flutes by using the above-proposed method. The resonance frequency calculation is applied at the first open side hole apart from the blow hole. In terms of the end admittance, the head-part (from the head to section *C*) and end-part (from section *I* to the end), are “fixed” on different play conditions, while the middle part consisting of the finger holes is “flexible”. Then, we take bass 6 as an example to introduce the frequency calculation method of the flutes.

In Fig. 5, we demonstrate the flute on condition of playing bass 6, whose fingering is that the first finger hole is open and the other finger holes are closed. By quoting Eqs. (4) and (5), one can get the inverse surface Green’s function of tube “*e*” and the end admittance at cross-section *B* straightforwardly. Looking at section *C*, we give the inverse surface Green’s function as

$$g_C^{-1} = \begin{pmatrix} -Y_0 \cot(kd_7) & \frac{Y_0}{\sin(kd_7)} \\ \frac{Y_0}{\sin(kd_7)} & -Y_0 \cot(kd_7) + Y_0' \tan(kt) + Y_B \end{pmatrix}. \quad (8)$$

The end admittance of such pipe at cross-section *C* ($Y_C = \frac{1}{g_C(1,1)}$) is a “fixed” value, which represents the end admittance of the head-part at cross-section *C*. By repeating the calculation process above, we can obtain the end admittance of left- and right-part at cross-section *H*, Y_H^L and Y_H^R , respectively. The end admittance of the first finger hole at cross-section *H* is $Y_H^{1st} = -Y_0' \cot(kt)$. With the resonance condition of Eq. (6), the resultant resonance condition of the flute playing bass 6 is given by

$$Y_H^L + Y_H^R + Y_H^{1st} = 0. \quad (9)$$

In a similar way, the resonance frequencies of other tones can also be obtained. The validity of our theoretical resonance frequency calculation method can be confirmed from the frequency spectra of the flute in Sec. 5. It should be noticed that special fingering called half-open is involved in some tones, such as the sixth finger hole of the alto 4. In this case, the special circumstance can be considered as two

identical parts, including one half open and another closed. Thus, the end admittance of the half-open side hole is $-\frac{Y'_0}{2} \cot(kt) + \frac{Y'_0}{2} \tan(kt)$.

4. Correction of the Geometrical Sizes of the Flute

Due to the end effect, the resultant resonance frequencies determined by the above resonant condition are markedly larger than the standard frequencies. Therefore, an appropriate end correction is needed for the pipe with an open end. For a finite tube, some investigations suggest that $0.6R$.¹² should be added at open outlet end of the unflanged pipe. Rayleigh gave $0.82R$ as the correction of a pipe with a large plane flanged end.¹³ If one makes the simplifying assumption that the flow pattern outside is exactly like that outside a hole in a thin plane sheet, the coefficient comes out at $\pi/4$,¹¹ which is very close to Rayleigh's result.

Considering the flute as a complex structure with many side holes, we need to modify the main pipe and side holes under a different boundary condition. As for the main tube corrections in length, different corrections are applied for different side hole opening-closing states. For the connecting tube “ i ” with geometrical length d_i and radius R , the correction is unnecessary if the side holes are both closed,

$$d_i^{\text{eff}} = d_i. \quad (10)$$

When one side hole is open, the effective length of the connecting tube should be corrected as

$$d_i^{\text{eff}} = d_i + \frac{\pi}{2} R. \quad (11)$$

As its both side holes are open, the effective length of the connecting tube “ i ” is

$$d_i^{\text{eff}} = d_i + \frac{\pi}{2} R + \frac{\pi}{2} R. \quad (12)$$

Take the fingering of bass 7, for example. For tube “4”, whose two side holes (the 3rd and 4th finger holes) are closed, the effective length of tube “4” is its real length $d_4^{\text{eff}} = d_4$. As one of the side hole of tube “3” (the 2nd finger hole) is open, the effective length thus has the form $d_3^{\text{eff}} = d_3 + \frac{\pi}{2} R$. For tube “2”, whose side holes (the 1st and 2nd finger holes) are both open, $d_2^{\text{eff}} = d_2 + \frac{\pi}{2} R + \frac{\pi}{2} R$. Special attention should be paid to the end tube “ a ”, where one of the ends is open to the external environmental. Since tube “ a ” takes a less important effect on the calculation of the resonance spectrum, we choose $\pi/4$ as the open outlet correction for unification. Thus, the effective length of the tube “ a ” is given by

$$d_a^{\text{eff}} = d_a + \frac{\pi}{2} R + \frac{\pi}{4} R. \quad (13)$$

For the correction of the side holes, we follow the experimental conclusions for references. Considering the open side hole as a circular aperture, we apply the

correction by measurement value,¹⁵

$$t^{\text{eff}} = t + 2r, \tag{14}$$

where r is the radius of the side hole, t is the thickness of the wall. For the base hole which is a side-by-side hole, we use the correction method given by Zhao⁶:

$$t^{\text{eff}} = \frac{1}{2} \times (t + 2.3r). \tag{15}$$

Additionally, in the fingerings of alto 4 (1) and high 4 (1), the sixth finger hole is half-open. In this case, we treat the sixth finger hole as an open hole when calculating the effective length of the main pipe with the side-hole correction being $t^{\text{eff}} = t + r$. The validity of our correction methods can be confirmed from Sec. 5.

5. The Resonance Frequency Spectra of the Flute in C

The geometrical sizes of the flute in C (data from the Internet flute maker) are shown in Table 1. Main pipe radius is R , and r denotes the radius of the side holes with r_b for the blow hole, t is responsible for the wall-thickness of the flute, which is the geometrical length of the side holes.

We know the surrounding environment has an effect on the sound frequencies of the flute. The relationship between the velocity of sound and temperature is $c = 331 + 0.6\theta^{\circ}\text{C}$.⁶ For the sake of calculation and without loss of reality, 15°C is selected as the ambient temperature in the following calculation. The velocity of sound is $c = 340\text{ m/s}$, and the air density is $\rho = 1.21\text{ kg/m}^3$.

The resonance frequencies of the flute are obtained by using the admittance matching method and precise correction principles. According to the resonance condition by Eq. (9), we show the resonance frequencies of bass 6 and alto 6 of the flute in C in Fig. 6(a). The horizontal coordinate represents the resonance frequency and the vertical coordinate represents total end admittance at cross-section H . The intersection dots generated by $Y = Y_H^L + Y_H^R + Y_H^{1st}$ and $Y = 0$ are resonance frequencies of bass 6 (377 Hz) and alto 6 (729 Hz). In this way, the resonant frequencies of the other tones can be obtained. Thus, we demonstrate the frequency spectrum of the flute in C in Fig. 6(b), where the resonance frequency calculation values of fingering (1), (2) with correction are shown compared to the standard values. It is found that the deviation caused by the calculation without correction is relatively considerable, especially among the high tone range. Obviously, our calculation

Table 1. Geometrical sizes of the flute in C.

The keys Of the flute	d_f (mm)	d_e (mm)	d_7 (mm)	d_6 (mm)	d_5 (mm)	d_4 (mm)	d_3 (mm)	d_2 (mm)	d_1 (mm)	d_c (mm)	d_b (mm)	d_a (mm)	R (mm)	r (mm)	r_b (mm)	t (mm)
C	10.00	107.00	81.90	32.30	36.40	47.80	15.60	49.00	75.30	25.00	15.00	25.00	10.50	5.00	5.50	4.00

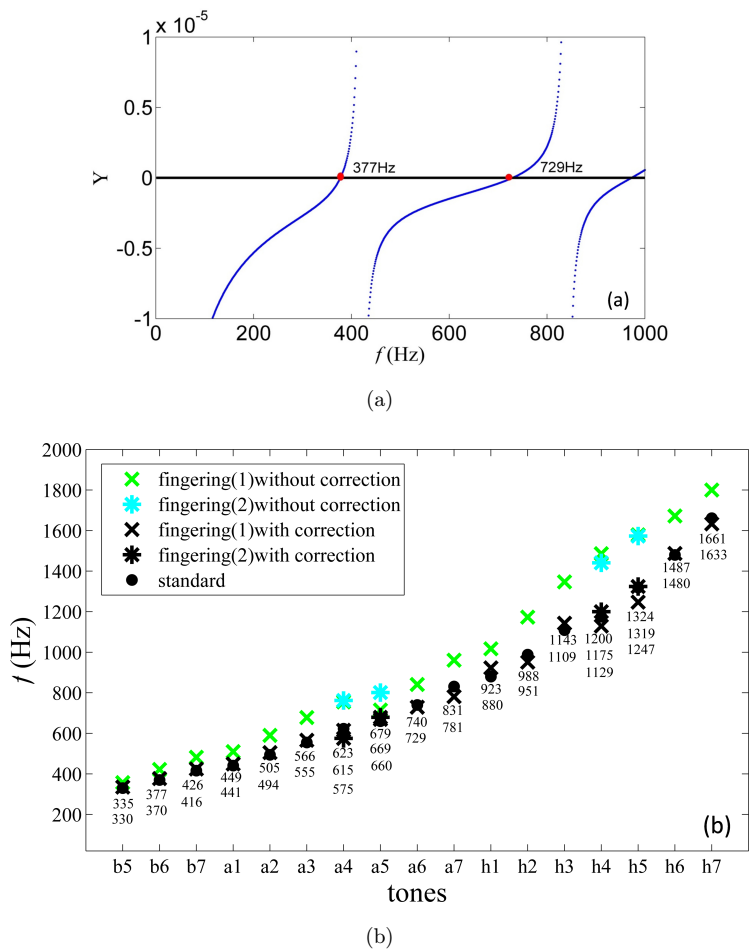


Fig. 6. (Color online) (a). The resonance frequencies of bass 6 (377 Hz) and alto 6 (729 Hz) of the flute in C . (b). The frequency spectrum of the flute in C . Forks ($\times$) represent the calculation frequencies for fingering (1). Stars ($*$) represent the calculation frequencies for fingering (2) of alto 4, alto 5, high 4 and high 5. Solid circles ($\bullet$) represent the standard frequencies of the tones.

results with correction are highly consistent with the experimental results for a wide tone range (bass 5 to high 7), which verified that the proposed correction is appropriate to calculate the resonance frequencies of the flutes.

6. Conclusion

In this paper, we theoretically calculated the resonance frequency spectra of the flute in C , by using the admittance matching method. We employed the inverse surface Green's function to calculate the end admittance for complex structures and provided the relationship between resonance frequency and admittance matching of

combined pipes which constitute a flute. The resonance frequencies are highly consistent with the experimental results with appropriate end corrections for main pipes as well as the side holes. Our results exhibit a wide range of tone (from bass 5 to high 7) with a minimized deviation to the standard value compared with earlier reported works. The proposed method is not only particularly effective to solve resonance frequency spectra of the flutes but also paves a way to study other lip-wind-instruments.

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